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Crossover between field theories with short-range and long-range exchange or correlations

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Abstract. The relation between scalar field theories with short-range and long-range exchange or correlations is studied. It is shown to all orders in perturbation theory that the critical exponents of fields and composite operators are continuous functions of the parameters $\alpha(\xi)$ characterising the decay rate of the long-range exchange (correlations) with power-like falloff $1/r^{d+2-2\alpha}(1/r^{d-2\xi})$. The scaling law $(d-2\xi)\nu=2$, where ν is the correlation length exponent, as well as the Harris criteria are shown to be exact to all orders in perturbation theory. A discrepancy between two widely used approaches to the crossover problem is pointed out.

1. Introduction

In several cases a local field-theoretic model, which we shall call the short-range (SR) model, is generalised by substituting a non-local operator for a local one. We shall call the resulting non-local model the long-range (LR) model. Typically, the Laplace operator is replaced by a non-local counterpart:

$$\int \mathrm{d}x\, \phi(x)(-\nabla^2)\phi(x) \to \int \mathrm{d}x\, \phi(x)(-\nabla^2)^{1-\alpha}\phi(x) \tag{1}$$

or, more correctly, in the momentum space

$$\int \frac{\mathrm{d}q}{(2\pi)^d}\, \phi(-q)q^2\phi(q) \to \int \frac{\mathrm{d}q}{(2\pi)^d}\, \phi(-q)q^{2(1-\alpha)}\phi(q) \tag{2}$$

where d is the dimension of space, and the same notation is used for both the field ϕ and its Fourier transform. This generalisation may be due to direct long-range exchange interaction, as in the case of a Heisenberg magnet [1], or long-range character of correlations of a random external field, as in the case of random magnets [2], in the field theory of turbulence [3] and diffusion in a random environment [4], or both these factors [5]. More formally, the substitution (2) may be used to regularise a field-theoretic model with ultraviolet divergences, leading to a special form of analytic regularisation [6].

In all cases, the problem of crossover between short-range and long-range models arises: formally, in the $\alpha\to 0$ limit of the LR model, the action of the SR model is recovered (at least in the momentum space). However, in this limit various quantities, which characterise the asymptotic behaviour of the LR model (the critical exponents in the first place), do not coincide with their counterparts in the SR model, leading at first sight to discontinuity of these quantities as functions of the parameter α. To be more definite, we shall consider a field theory of a scalar n-component field ϕ with the O(n)-symmetric action

$$S(\phi)=-\tfrac{1}{2}a\nabla\phi\nabla\phi-\tfrac{1}{2}\tau\phi^2-\tfrac{1}{2}b\phi(-\nabla^2)^{1-\alpha}\phi-\tfrac{1}{24}\lambda(\phi^2)^2 \tag{3}$$

where the values of parameters $a>0$, $\tau\geqslant 0$ and $\lambda>0$ correspond to the symmetric phase with zero expectation value of the field ϕ. In equation (3), and all subsequent similar formulae, necessary integrals and sums are implied. The Green functions of the theory are calculated as functional averages with the weight $\exp S(\phi)$.

It was pointed out some time ago by Sak [7] that for this model the discontinuities in α are removed if the competition between q^2 and $q^{2(1-\alpha)}$ terms in the $\alpha\to 0$ limit of the LR model is properly taken into account. In the renormalisation scheme of Wilson [8], local terms $\propto q^2$ are always being produced by renormalisation, even if the initial action contains only the non-local term. For finite α, the corresponding local terms of the renormalised action are irrelevant at the critical point, but they become at least marginal in the $\alpha\to 0$ limit, and therefore their effect should be taken into account. It was conjectured by Sak [7] that this interplay removes the discontinuity of exponents as functions of α and that for long-range exchange with $0\leqslant\alpha<\eta/2$ the critical exponents actually have the short-range values, and at the value $\alpha=\eta/2$ they coincide up to logarithmic corrections (η is the Fisher exponent of the short-range ϕ^4 model). Sak showed this to be the case in the first non-trivial order of perturbation theory (i.e. to the order $O(\varepsilon^2)$ in the ε expansion) using the recursion relations of Wilson. Further support for this assertion was provided by the result of Suzuki [9], from which it follows that this continuity holds to the order $O(1/n)$ in the $1/n$ expansion for arbitrary space dimensionality $2<d<4$. Although this is generally believed to be true to all orders in perturbation theory [1], it probably cannot be proved with these methods.

On the other hand, if the LR model is used as the regularised version of the SR model, then the upper critical dimension becomes α dependent:

$$d_c=4-4\alpha$$

and expansions of critical quantities in $\tilde{\varepsilon}=4-4\alpha-d$ may be constructed, in full analogy to the usual $4-d=\varepsilon$ expansion. The 'physical' value of $\tilde{\varepsilon}$ then corresponds to the initial short-range model, i.e. $\tilde{\varepsilon}_{\mathrm{ph}}=4-d$. The long-range term, however, is not renormalised since it is non-analytic in q^2, and renormalisation yields only terms polynomial in momenta. This fact is not totally trivial and will be explained in detail in the next section. As a consequence of this, the anomalous dimension of the field ϕ is zero, which leads to a false result for the Fisher exponent: $\eta=0$. This problem is the mentioned discontinuity problem put in other words, and the cure is the same: the 'dangerous' irrelevant operator $\phi\nabla^2\phi$ must be included in the renormalisation procedure in the $\alpha\to 0$ limit of the LR model.

The purpose of this paper is to present a relatively simple treatment of this problem to all orders in perturbation theory. Our approach is similar to that of Weinrib and Halperin [2] and we generalise some of their results, but we have used the field-theoretic framework, in which the properties of different renormalisation schemes and their relations have been analysed in great detail. We also point out that the 'standard' approach [10, 11] to fixed-point stability problems, which exploits renormalisation of composite operators, seems to fail in this case, contrary to the problem of diffusion in random environments, in which both approaches complement each other [12]. The paper is organised as follows: in § 2 we describe the renormalisation procedure for the ϕ^4 model with both short- and long-range exchange. Renormalisation constants are constructed explicitly in § 3, in which the stability of non-trivial fixed points of the corresponding renormalisation group equations is also studied and the field dimension is shown to be a continuous function of α in perturbation theory. In § 4 we consider the ϕ^4 model with both short-range and long-range correlated disorder, and prove the scaling law $(d-2\xi)\nu = 2$ [2] and the Harris criteria [2, 13] to be exact to all orders in perturbation theory. Section 5 is devoted to renormalisation of composite operators and the failure of the 'standard' approach in the problem of crossover between short-range and long-range exchange.

2. Scalar ϕ^4 model with short-range and long-range exchange

We shall treat here the interplay of short-range and long-range exchange within the field-theoretic renormalisation group approach to critical phenomena [11]. To this end, we consider a Euclidean field theory with the following basic action (we follow the terminology of Collins [14]), which includes both the short-range and long-range term:

$$S(\phi) = -\tfrac{1}{2}\nabla\phi\nabla\phi - \tfrac{1}{2}b\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\phi^4 \tag{4}$$

where b and λ are the renormalised parameters of the model. For convenience, the 'mass' term has been omitted (i.e. the model is investigated at the critical point) and the coefficient a of the short-range term (3) has been absorbed in a redefinition of the field and the remaining parameters. The Green functions of the theory are calculated as functional averages of field monomials with the weight $\exp(S)$. In order to construct a tractable diagrammatic expansion, one of the quadratic terms has to be treated as an interaction term, while the other determines the bare propagator of the model. We shall regard the long-range term as an interaction. We shall use dimensional regularisation with minimal subtractions thus introducing a small parameter $\varepsilon = 4-d$. Since we are ultimately interested in small values of α, it is natural to consider this small parameter on equal grounds with ε. The effect of the long-range 'interaction' term is to shift the powers of momenta in propagators by a multiple of α. Therefore, we arrive at a diagrammatic expansion of the usual local ϕ^4 field theory, which has effectively been regularised by a combination of dimensional and analytic regularisation schemes. Due to this, the usual minimal subtraction scheme has to be slightly modified. Instead of poles in ε, ultraviolet divergences show in the form of poles in parameters $\delta = p\varepsilon + r\alpha$, where p and r are rational numbers, and this leads to a modification of the standard relations between anomalous dimensions and renormalisation constants [15], details of which will be given in the next section. We introduce the usual scale-setting parameter μ in order to make the coupling constants dimensionless:

$$S = -\tfrac{1}{2}\nabla\phi\nabla\phi - \tfrac{1}{2}b\mu^{2\alpha}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^{\varepsilon}\phi^4. \tag{5}$$

Standard power counting shows that this action is multiplicatively renormalisable. However, the momentum dependence of the long-range term is not analytic, and therefore it is not renormalised, since the counterterms arising from the renormalisation procedure are polynomial functions of momenta.

This can be seen as follows. The long-range term effectively produces new lines corresponding to 'propagators' of the form $q^{2(1-\alpha)}$ in our case, when the long-range term is treated as an interaction. Apart from this, an effective two-point 'vertex' appears in the chains consisting of the new and original lines ($\propto 1/q^2$). Thus, the diagrammatic rules are standard, except for the analytic form of the new lines. However, upon the parametrisation

$$q^{2(1-\alpha)} = \frac{1}{\Gamma(\alpha-1)} \int_0^\infty \mathrm{d}s\, s^{\alpha-2} \exp(-sq^2)$$

the α dependence is transferred to the parametric integral and is of the same type as in the analytic renormalisation scheme [6], whereas the momentum integrals take the familiar Gaussian form. This means that all the standard machinery [14, 16] of the renormalisation theory may be applied, and the primitive convergence or divergence of each (one-particle irreducible, 1PI) graph Γ of the theory is determined as usual by the degree of divergence δ_Γ of the graph, which in this case may be an irrational number. By definition, the degree of divergence is equal to the dimension of the 1PI graph in the momentum space. It is a fundamental fact of the theory of renormalisation that a graph with a negative degree of divergence is convergent after the subtraction of its subdivergences (which correspond to divergent subgraphs).

Proceeding in the usual way [14], we consider a 1PI graph Γ with non-negative degree of divergence $\delta_\Gamma \geqslant 0$. Let $R(\Gamma)$ be the renormalised (finite) value of the graph and denote by $\bar{R}(\Gamma)$ the value of the graph with subtracted subdivergences. Then

$$R(\Gamma) = \bar{R}(\Gamma) - T_\Gamma \bar{R}(\Gamma) \tag{6}$$

where T_Γ is an operator, which extracts the divergence of $\bar{R}(\Gamma)$. Differentiating with respect to external momenta, we arrive at a similar relation for the graph $\partial_q^l \Gamma$, where ∂_q^l denotes any l-fold differentiation with respect to the external momenta. Choosing $l > d_\Gamma$ we ensure that the differentiated graph has a negative degree of divergence. Therefore, after subtraction of subdivergences it yields a finite expression, i.e. $\bar{R}(\partial_q^l \Gamma)$ does not contain divergences, and the corresponding counterterm (the second term in the right-hand side of the analogue of the relation (6)) may be chosen to be finite. From this it immediately follows that the counterterm for the initial graph Γ is polynomial in the external momenta. If it were not, the divergences in it would be transferred to the analogue of (6) for the differentiated graph $\partial_q^l \Gamma$, thus leading to a contradiction. For this argument it is essential that the contribution of the non-local term of the action (5) may be represented in terms of effective new lines and *local* vertices, which allows us to use the standard techniques of renormalisation theory.

Adding the relevant (local) counterterms, we obtain the renormalised action in the form

$$S = -\tfrac{1}{2} Z_\phi \nabla\phi\nabla\phi - \tfrac{1}{2} b\mu^{2\alpha} \phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^\varepsilon Z_1 \phi^4. \tag{7}$$

This action leads to the following connection between renormalised and bare parameters:

$$b \to b_0 = b\mu^{2\alpha} Z_\phi^{-1} \qquad \lambda \to \lambda_0 = \lambda\mu^\varepsilon Z_1 Z_\phi^{-2}. \tag{8}$$

The definition of beta functions is standard [11]:

$$\beta_e = \mu \frac{\partial}{\partial\mu}\bigg|_0 e \qquad e = (b, \lambda) \tag{9}$$

and leads to the expressions

$$\beta_b = b[-2\alpha + \gamma_\phi(b, \lambda)]$$
$$\beta_\lambda = \lambda[-\varepsilon + \gamma_1(b, \lambda) + 2\gamma_\phi(b, \lambda)] \tag{10}$$

where

$$\gamma_1 = -\mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_1 \qquad \gamma_\phi = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_\phi. \tag{11}$$

In the definitions (9) and (11) derivatives are taken with fixed values of the bare parameters b_0 and λ_0. Stability of the fixed point (b_*, λ_*) defined by $\beta_b(b_*, \lambda_*) = \beta_\lambda(b_*, \lambda_*) = 0$ is characterised by the eigenvalues of the matrix B of the first derivatives of the beta functions at the fixed point:

$$B = \begin{pmatrix} \partial\beta_b/\partial b & \partial\beta_b/\partial\lambda \\ \partial\beta_\lambda/\partial b & \partial\beta_\lambda/\partial\lambda \end{pmatrix}_{b=b_*, \lambda=\lambda_*}. \tag{12}$$

We have now described the renormalisation scheme, and to proceed further we need some information about the structure of renormalisation constants as functions of b and λ.

3. Renormalisation and stability of fixed points

We need the following general result, which can readily be obtained as a combination of two well known theorems [16] describing the structure of divergences in analytic and dimensional renormalisation schemes: if these approaches are being used simultaneously, as in our case, then the divergences of a given one-particle-irreducible (1PI) graph Γ_I with I arbitrarily numbered lines appear in the form of a polynomial of the quantities $1/\Delta_{p,l}$, where $\Delta_{p,l}$ are of the form

$$\Delta_{p,l} = \sum_{j=1}^{l} \rho_{p(j)} + \tfrac{1}{2}L_l\varepsilon - N_{p,l}. \tag{13}$$

Here, the set of $l \leqslant I$ numbers $1 \leqslant p(j) \leqslant I$ denotes a divergent subgraph Γ_l of the graph Γ_I, spanned by the lines with these numbers. Through L_l we denote the number of loops in the subgraph Γ_l, ε is the deviation of space dimensionality d from the upper critical dimension d_c: $\varepsilon = d_c - d$, and $N_{p,l}$ is an integer not exceeding one-half of the value of the degree of divergence δ_{Γ_l} of the subgraph Γ_l. By definition, δ_{Γ_l} is the dimension of the graph Γ_l in the momentum space. Parameters $\rho_{p(j)}$ regularise the propagators corresponding to the lines of the graph in the following way:

$$\mathrm{reg}_{p(j)}\left(\frac{1}{q^2 + m_{p(j)}^2}\right) = \frac{1}{(q^2 + m_{p(j)}^2)^{1+\rho_{p(j)}}} \tag{14}$$

where the mass parameters $m_{p(j)}$ have been introduced in order to avoid infrared divergences (inequalities $\mathrm{Re}\,\rho_{p(j)} > 0$ are assumed to hold, since this scheme has originally been constructed to handle ultraviolet divergences). In our case the ρ are multiples of $\alpha > 0$ and thus this condition is fulfilled. Moreover, small parameters of the problem are of the form $\delta = p\varepsilon + r\alpha$. Therefore only the poles with $N_{p,l} = 0$ are relevant. In order to obtain the contribution of a 1PI graph to renormalisation constants, the terms non-analytic in ρ and ε corresponding to divergent subgraphs have to be subtracted. In a subtraction scheme of the minimal type, only terms linear in $1/\Delta$

contribute to the anomalous dimensions. For each 1PI graph only one term linear in $1/\Delta$ with $N_{p,l}=0$ remains, for which $\Delta_{p,l}=\Sigma_{j=1}^{l}\rho_j+L_l\varepsilon/2$. Therefore, for the theory without the long-range term the renormalisation constants may be expressed in the form

$$Z_{\phi\mathrm{SR}}=1+\sum_{n=2}\frac{\lambda^n A_n}{n\varepsilon}+\mathrm{O}(\varepsilon^{-2})$$
$$Z_{1\mathrm{SR}}=1+\sum_{n=1}\frac{\lambda^n B_n}{n\varepsilon}+\mathrm{O}(\varepsilon^{-2}) \tag{15}$$

where the sums are taken over the number of loops in 1PI self-energy and four-point vertex graphs, respectively. For functions $\gamma_{1\mathrm{SR}}$ and $\gamma_{\phi\mathrm{SR}}$ one then obtains the expansions

$$\gamma_{\phi\mathrm{SR}}(\lambda)=-\sum_{n=2}\lambda^n A_n$$
$$\gamma_{1\mathrm{SR}}(\lambda)=\sum_{n=1}\lambda^n B_n. \tag{16}$$

It is not difficult to see that the effect of the insertions of the LR term is to replace each propagator in the graphs by a sum of analytically regularised propagators in the fashion

$$\frac{1}{q^2+m^2}\to\sum_{l=0}^{\infty}\frac{(-b)^l q^{2(l-\alpha)}}{(q^2+m^2)^{1+l}}. \tag{17}$$

The effective propagators generated by this formula differ from those of the 'canonical' form of analytic regularisation (14). Due to the independence [11] of renormalisation constants of the infrared regularisation (the parameters m in our case), this does not affect the critical properties of the theory. As a result of the substitution (17), each term in the sums (15) will be replaced by a sum of terms with denominators shifted by a multiple of α and the expressions for renormalisation constants take the form

$$Z_\phi=1+\sum_{n=2}\sum_{l=0}\binom{1-2n}{l}\frac{b^l\lambda^n A_n K_{nl}(\varepsilon,\alpha)}{n\varepsilon+2l\alpha}+\mathrm{O}(\delta^{-2})$$
$$Z_1=1+\sum_{n=1}\sum_{l=0}\binom{-2n}{l}\frac{b^l\lambda^n B_n K'_{nl}(\varepsilon,\alpha)}{n\varepsilon+2l\alpha}+\mathrm{O}(\delta^{-2}) \tag{18}$$

where the functions $K\to 1$, when $\varepsilon\to 0$, $\alpha\to 0$. These expressions are valid in the case of independent small parameters α and ε. However, dimensional considerations suggest (and this is confirmed by the result of Sak [7]) that the crossover from the LR to the SR regime sets in at $\alpha=\mathrm{O}(\varepsilon)$ or smaller. In this case, minimal subtractions correspond to the choice $K_{nl}=K'_{nl}=1$, which we shall adopt. Let us introduce the notation $[Z_i]_1$ for those parts of the renormalisation constants Z_i which consist of the terms containing simple poles in the parameters $\delta=l\varepsilon+2k\alpha$, where l and k are positive integers. Thus, for instance, $[Z_\phi]_1$ stands for the sum

$$[Z_\phi]_1=\sum_{n=2}\sum_{l=0}\binom{1-2n}{l}\frac{b^l\lambda^n A_n}{n\varepsilon+2l\alpha}.$$

Functions γ_ϕ and γ_1 are expressed through the renormalisation constants as follows:

$$\gamma_\phi=-\varepsilon\lambda\frac{\partial}{\partial\lambda}[Z_\phi]_1-2\alpha b\frac{\partial}{\partial b}[Z_\phi]_1$$
$$\gamma_1=\varepsilon\lambda\frac{\partial}{\partial\lambda}[Z_1]_1+2\alpha b\frac{\partial}{\partial b}[Z_1]_1. \tag{19}$$

Substituting the expansions (18) into this equation we obtain

$$\begin{aligned}\gamma_\phi(b,\lambda) &= -\sum_{n=2}\sum_{l=0}\binom{1-2n}{l} b^l\lambda^n A_n = -(1+b)\sum_{n=2}\lambda^n(1+b)^{-2n}A_n \\ \gamma_1(b,\lambda) &= \sum_{n=1}\sum_{l=0}\binom{-2n}{l} b^l\lambda^n B_n = \sum_{n=1}\lambda^n(1+b)^{-2n}B_n.\end{aligned} \tag{20}$$

Comparison with the relations (16) then reveals that the γ here are given by essentially the same functions as in the SR case (19), the main difference being in the arguments of these functions: λ of the SR case is replaced by $\lambda(1+b)^{-2}$ in (20):

$$\gamma_\phi(b,\lambda) = (1+b)\gamma_{\phi\mathrm{SR}}[\lambda(1+b)^{-2}] \qquad \gamma_1(b,\lambda) = \gamma_{1\mathrm{SR}}[\lambda(1+b)^{-2}]. \tag{21}$$

The beta functions (10) therefore take the form

$$\begin{aligned}\beta_b &= b\{-2\alpha + (1+b)\gamma_{\phi\mathrm{SR}}[\lambda(1+b)^{-2}]\} \\ \beta_\lambda &= \lambda\{-\varepsilon + \gamma_{1\mathrm{SR}}[\lambda(1+b)^{-2}] + 2(1+b)\gamma_{\phi\mathrm{SR}}[\lambda(1+b)^{-2}]\}.\end{aligned} \tag{22}$$

The effective expansion parameter turned out to be $\lambda(1+b)^{-2} = g$ instead of the initial coupling constant λ. Therefore we shall study the properties of fixed points in terms of b and g rather than b and λ. For b and g, the beta functions are the following:

$$\begin{aligned}\beta_b &= b[-2\alpha + (1+b)\gamma_{\phi\mathrm{SR}}(g)] \\ \beta_g &= g\left(-\varepsilon + \frac{4\alpha b}{1+b} + \gamma_{1\mathrm{SR}}(g) + 2\gamma_{\phi\mathrm{SR}}(g)\right).\end{aligned} \tag{23}$$

The corresponding renormalisation group equations have two non-trivial fixed points and we shall investigate first the conditions of stability of the fixed point corresponding to the SR model, at which $b_* = 0$ and g_* is given by the equation

$$\gamma_{1\mathrm{SR}}(g_*) + 2\gamma_{\phi\mathrm{SR}}(g_*) - \varepsilon = 0. \tag{24}$$

At this fixed point, the matrix of first derivatives B turns out to be triangular: $B_{bg}^{\mathrm{SR}} = 0$, and the eigenvalues coincide with the diagonal elements of the matrix B. The lower diagonal element B_{gg}^{SR} is positive, at least for small $\varepsilon > 0$, since the SR fixed point is perturbatively infrared stable [11]. The upper diagonal element B_{bb} is of the form

$$B_{bb}^{\mathrm{SR}} = \gamma_{\phi\mathrm{SR}}(g_*) - 2\alpha$$

and due to the relation between the Fisher exponent η and the anomalous field dimension γ_ϕ:

$$\eta = \gamma_{\phi\mathrm{SR}}(g_*)$$

it follows that the SR fixed point is stable against the long-range perturbation, provided the condition of Sak [7]

$$\eta > 2\alpha \tag{25}$$

is fulfilled. This result may be obtained also from a simple scaling argument [5] which, however, does not apply to the case of the LR fixed point with $b = \bar{b} \neq 0$, $g = \bar{g} \neq 0$.

This fixed point is determined by equations

$$(1+\bar{b})\gamma_{\phi\mathrm{SR}}(\bar{g}) - 2\alpha = 0 \tag{26}$$

$$\gamma_{1\mathrm{SR}}(\bar{g}) + 2\gamma_{\phi\mathrm{SR}}(\bar{g}) - \varepsilon + \frac{4\alpha\bar{b}}{1+\bar{b}} = 0. \tag{27}$$

Using equation (26), one may exclude the parameter $\bar{b}$ from equation (27) to obtain

$$\gamma_{1\mathrm{SR}}(\bar{g}) - \varepsilon + 4\alpha = 0 \tag{28}$$

the fixed-point equation of the 'purely LR' model, i.e. the model defined by the basic action of the form

$$S(\phi) = -\tfrac{1}{2}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{2}\tau\phi^2 - \tfrac{1}{24}\lambda(\phi^2)^2.$$

The lower diagonal matrix element B_{gg}^{LR} is given by the same function of g as in the short-range case, taken at the point $\bar{g}$ instead of g_*. Thus, for small α, this matrix element is positive. For the upper diagonal matrix element and the determinant we obtain

$$\begin{aligned} B_{bb}^{\mathrm{LR}} &= \bar{b}\gamma_{\phi\mathrm{SR}}(\bar{g}) = \frac{2\bar{b}\alpha}{1+b} \\ \det B^{\mathrm{LR}} &= \frac{2\alpha\bar{g}\bar{b}}{1+\bar{b}} \frac{\mathrm{d}}{\mathrm{d}g}\bigg|_{g=\bar{g}} \gamma_{1\mathrm{SR}}(g). \end{aligned} \tag{29}$$

Both expressions are perturbatively positive for $\bar{b} > 0$. These relations show that this fixed point is unstable when $\bar{b} < 0$. The sign of $\bar{b}$ is determined by the ratio of the parameter α and the function $\gamma_{\mathrm{SR}}(g)$ taken at the LR fixed point $g = \bar{g}$, which we denote through $\bar{\eta} = \gamma_{\phi\mathrm{SR}}(\bar{g})$ (note that the actual anomalous field dimension in the LR case is equal to $(1+\bar{b})\bar{\eta}$ and it is determined by the fixed-point equation (26)). The fixed-point equations (26) and (27) yield

$$\bar{b} = \frac{2\alpha}{\bar{\eta}} - 1 \tag{30}$$

which, with (29), leads to the following condition of stability of the LR fixed point:

$$\bar{\eta} < 2\alpha. \tag{31}$$

Summarising, we conclude that the long-distance behaviour of Green functions of the field theory (5) is governed by the power-like behaviour of the long-range term for $\alpha > \eta/2$, while in the opposite case $\alpha < \eta/2$ it is determined by the Fisher exponent η of the short-range model. In the borderline case both regimes lead to the same asymptotic behaviour, up to logarithmic corrections.

4. ϕ^4 model with long-range correlated quenched disorder

A similar treatment may be carried out for the ϕ^4 model with a quenched disorder field with competing short-range and long-range correlations. This case has been analysed earlier by Weinrib and Halperin [2] in the leading order of perturbation theory and we generalise some of their results to arbitrary order. We begin with a theory of two fields ϕ and φ with the basic action

$$\begin{aligned} S(\phi, \varphi) = &-\tfrac{1}{2}\nabla\phi^a\nabla\phi^a - \tfrac{1}{2}\tau(\phi^a)^2 - \tfrac{1}{24}\lambda((\phi^a)^2)^2 \\ &+ \tfrac{1}{24}g_1(\phi^a)^2(\phi^b)^2 - (1/2g_2)\varphi(-\nabla^2)^{\xi}\varphi - \tfrac{1}{2}\varphi(\phi^a)^2. \end{aligned} \tag{32}$$

Here, the usual replica trick has been used to produce N copies of the main field ϕ labelled by a and b (the corresponding sums over these indices as well as the usual limit $N \to 0$ are implied), and the new four-point term with coupling constant g_1

corresponds to the short-range part of the quenched disorder, whereas the φ field with the three-point interaction term introduces the long-range part of the disorder. We want to deal with field theories with local interactions only and therefore have not integrated out the field φ. Upon renormalisation, the renormalisation constants appear as follows (we again omit the mass term):

$$S(\phi, \varphi) = -\tfrac{1}{2}Z_\phi \nabla\phi^a \nabla\phi^a - \tfrac{1}{24}\lambda\mu^\varepsilon Z_1((\phi^a)^2)^2 + \tfrac{1}{24}g_1\mu^\varepsilon Z_3(\phi^a)^2(\phi^b)^2$$
$$- (1/2g_2\mu^\delta)\varphi(-\nabla^2)^\xi\varphi - \tfrac{1}{2}Z_4\varphi(\phi^a)^2 \tag{33}$$

where we have introduced the scaling factor μ, scaled the field $\varphi \to \mu^{-\delta/2}\varphi$, and denoted $\delta = \varepsilon + 2\xi$ ($\varepsilon = 4 - d$). Strictly speaking, the action (32) is multiplicatively renormalisable only in the limit $N \to 0$, in which the counterterms quadratic in φ vanish, since all the graphs in the diagrammatic expansion of the full $\varphi\varphi$ propagator contain at least one closed loop $\propto N$ of the main field propagators ($\propto \delta_{ab}$ in the replica space). Using the standard connection between bare and renormalised parameters:

$$\lambda_0 = \lambda\mu^\varepsilon Z_1 Z_\phi^{-2} \qquad g_{10} = g_1\mu^\varepsilon Z_3 Z_\phi^{-2} \qquad g_{20} = g_2\mu^\delta Z_4^2 Z_\phi^{-2} \tag{34}$$

we arrive at the set of beta functions

$$\begin{aligned} \beta_\lambda &= \lambda[-\varepsilon + \gamma_1 + 2\gamma_\phi] \\ \beta_{g1} &= g_1[-\varepsilon + \gamma_3 + 2\gamma_\phi] \\ \beta_{g2} &= g_2[-\delta + 2\gamma_4 + 2\gamma_\phi] \end{aligned} \tag{35}$$

where the γ are defined as before (11). Note that the definition of g_2 implies the following renormalisation of the field φ:

$$\varphi \to Z_4 Z_\phi^{-1}\varphi = Z_\varphi^{1/2}\varphi. \tag{36}$$

The procedure of the preceding section may be carried out in a similar fashion with the same results about the continuity of anomalous dimensions as functions of ξ and we shall not repeat it here. Rather, we would like to point out that it also allows for an all-order proof of the new scaling law of Weinrib and Halperin [2], which in our notation has the form

$$\nu = \frac{2}{d - 2\xi} = \frac{2}{4 - \delta} \tag{37}$$

where ν is the correlation length exponent in the 'mixed' disorder regime (i.e. when both the short-range and the long-range disorder affect the critical behaviour of the model). A scaling argument based on the extended Harris criterion [2] supports the conjecture that this relation is exact. To prove it in perturbation theory, let us consider the equations which determine the 'mixed' fixed point $\lambda_* \neq 0$, $g_{1*} \neq 0$ and $g_{2*} \neq 0$ for this model (the 'long-range disorder' fixed point in [2]):

$$\begin{aligned} &-\varepsilon + \gamma_1 + 2\gamma_\phi = 0 \\ &-\varepsilon + \gamma_3 + 2\gamma_\phi = 0 \\ &-\delta + 2\gamma_4 + 2\gamma_\phi = 0. \end{aligned} \tag{38}$$

Using the standard relation [11] between the exponent ν and the anomalous dimension of the composite operator ϕ^2:

$$\gamma_{\phi^2}(\lambda_*, g_{1*}, g_{2*}) = 2 - 1/\nu \tag{39}$$

and the fact that the anomalous dimension γ_{ϕ^2} may be expressed through other γ as

$$\gamma_{\phi^2} = \gamma_4 + \gamma_\phi \tag{40}$$

we immediately see that the last fixed-point equation (38) determines the exponent ν exactly to all orders in perturbation theory in accordance with the scaling law (37). This phenomenon is familiar from other models with long-range correlations [3, 12].

In this scheme, both the usual [13] and the extended [2] Harris criterion are recovered as stability conditions of the pure SR fixed point. To see this, let us consider the following diagonal matrix elements of the B matrix (analogous to (12)) at the pure SR fixed point $\lambda_* \neq 0$, $g_{1*} = 0$ and $g_{2*} = 0$:

$$\begin{aligned} \partial_{g_1}\beta_{g_1}(\lambda_*, 0, 0) &= -\varepsilon + \gamma_3(\lambda_*, 0, 0) + 2\gamma_\phi(\lambda_*, 0, 0) \\ \partial_{g_2}\beta_{g_2}(\lambda_*, 0, 0) &= -\delta + 2\gamma_4(\lambda_*, 0, 0) + 2\gamma_\phi(\lambda_*, 0, 0). \end{aligned} \tag{41}$$

At this fixed point the B matrix is triangular. Therefore positivity of these quantities guarantees the stability of the pure SR fixed point with respect to disorder effects. To relate this condition to the Harris criteria, let us analyse diagrammatic contributions to the vertex renormalisation constants. We shall identify the four-point vertices of the action (32) by the corresponding coupling constants. We use the fact that the γ in the relations (41) are calculated for $g_1 = g_2 = 0$. In this case only graphs without three-point and g_1 vertices contribute to Z_ϕ and Z_1, whereas graphs which contribute to Z_3 and Z_4 contain one g_1 and one three-point vertex, respectively. For Z_4, this three-point vertex is obviously an external one (i.e. external lines emanate from it) with one external φ line and two internal ϕ lines. In the case of four-point vertex renormalisation, graphs containing one external g_1 vertex with one external ϕ line have a 'local' structure in replica indices, $\propto \delta_{ab}\delta_{ac}\delta_{ad}$, and therefore contribute to the renormalisation constant Z_1 only. Most of the graphs with one internal g_1 vertex also have the same 'local' structure and do not affect Z_3. There is, however, a class of graphs with one internal g_1 vertex, which contribute to Z_3: these graphs consist of two blocks connected only at the g_1 vertex (i.e. if this vertex is removed, the graph becomes disconnected). Due to this, they give rise to terms at least quadratic in $1/(p\varepsilon + r\xi)$ and do not affect γ_3, which we are ultimately interested in. Thus, we are effectively left with four-point graphs containing one external g_1 vertex with two external and two internal ϕ lines. It is not difficult to see that, apart from this external vertex, these graphs coincide with those of the three-point vertex renormalisation. The four-point graphs, however, have an extra symmetry factor of 2, owing to which the contribution of the four-point graphs to γ_3 is twice as large as the contribution of the three-point graphs to γ_4. Thus, we obtain

$$\gamma_3(\lambda_*, 0, 0) = 2\gamma_4(\lambda_*, 0, 0). \tag{42}$$

When the relations (39), (40) and (42) are taken into account, the requirement of positivity of the matrix elements (41), depending on the value of ξ, leads either to the usual Harris criterion [13]:

$$d\nu - 2 > 0 \qquad \xi < 0 \tag{43}$$

or to the extended Harris criterion [2]:

$$\nu(d - 2\xi) - 2 > 0 \qquad \xi > 0 \tag{44}$$

where ν is the correlation length exponent of the pure short-range model. These criteria refer to stability of the pure short-range regime with respect to disorder, but it is not difficult to see that the extended Harris criterion may be generalised to characterise stability of the short-range disorder regime against the long-range disorder. The same inequality (44) (but without the condition $\xi > 0$)

$$\nu(d - 2\xi) - 2 > 0 \tag{45}$$

ensures this stability. It should be kept in mind, however, that in formulae (37), (43), (44) and (45) the correlation length exponent ν (being a continuous function of parameters ε and ξ) assumes different values corresponding to different scaling regimes.

5. Dimensions of composite operators

In this section, to complete the analysis of continuity of critical exponents as functions of α (or ξ), we shall deal with the behaviour of anomalous dimensions of composite operators. We shall consider explicitly only the simplest case of the operator $\phi^2(x)$ in the long-range exchange model (4), since generalisation of its treatment to other composite operators is obvious.

The Green functions with ϕ^2 insertions are generated by the action

$$S = -\tfrac{1}{2}Z_\phi \nabla\phi\nabla\phi - \tfrac{1}{2}b\mu^{2\alpha}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^\varepsilon Z_1\phi^4 - \tfrac{1}{2}Z_2 t\phi^2 \tag{46}$$

as functional derivatives with respect to the source field $t = t(x)$. We are ultimately interested in the case when the source field $t(x)$ of composite operator ϕ^2 is a constant, but to avoid infrared problems in massless theory one has to introduce the source in this form [11]. Most conveniently, the new renormalisation constant Z_2 may be extracted from a 1PI two-point function with one ϕ^2 insertion: $\Gamma_{\phi^2\phi\phi}$. Corresponding graphs are of the same type as for the vertex renormalisation constants, and therefore one readily obtains the relation

$$\gamma_2(b, \lambda) = \gamma_{2\mathrm{SR}}(g) \tag{47}$$

where γ_2 is defined by

$$\gamma_2 = -\mu \frac{\partial}{\partial\mu}\bigg|_0 \ln Z_2 \tag{48}$$

and $\gamma_{2\mathrm{SR}}$ denotes the counterpart of this function in the SR case. At the SR fixed point one arrives at the standard result [11]

$$1/\nu = 2 - \gamma_{2\mathrm{SR}}(g_*) - \gamma_{\phi\mathrm{SR}}(g_*) = 2 - \gamma_{2\mathrm{SR}}(g_*) - \eta \tag{49}$$

which at the LR fixed point is replaced by

$$1/\nu = 2 - \gamma_{2\mathrm{SR}}(\bar{g}) - (1 + \bar{b})\gamma_{\phi\mathrm{SR}}(\bar{g}) = 2 - \gamma_{2\mathrm{SR}}(\bar{g}) - 2\alpha. \tag{50}$$

Due to the fact that $\bar{g} \to g_*$ as $\bar{b} \to 0$ (27), continuity of the exponent ν is obvious. This is obviously true also for the anomalous dimensions of all the other composite operators, since the dimensions by construction are expressed as regular expansions in g and b.

Finally, we would like to point out a discrepancy between this approach and the alternative approach [10, 11] to the stability of fixed points with respect to perturbation by irrelevant operators. In the latter scheme the perturbation is renormalised as an irrelevant composite operator and its anomalous dimension is calculated, after which it can be checked whether the total dimension would render this operator marginal or relevant. In our case, however, this method does not work. There are no formal problems if we regard the long-range part of the action (4) as a perturbation: results are the same as in the preceding section. In the opposite case, however, problems do arise. To see this, let us consider the renormalised action

$$S=-\tfrac{1}{2}\phi(-\nabla^2)^{1-\alpha}\phi-\tfrac{1}{24}\lambda\mu^{\tilde{\varepsilon}}Z_1\phi^4-\tfrac{1}{2}tZ_2\nabla\phi\nabla\phi \tag{51}$$

for which the upper critical dimension $d_c=4-4\alpha$ and $\tilde{\varepsilon}=d_c-d$. There is no need to introduce coordinate-dependent sources for the operator $\nabla\phi\nabla\phi$, since the derivatives in this operator prevent infrared divergences. At one-loop order we obtain the following expression for the beta function:

$$\beta_\lambda=\lambda\left(-\tilde{\varepsilon}+\frac{n+8}{6}C_\alpha\lambda\right) \tag{52}$$

where (Γ is the gamma function)

$$C_\alpha=\frac{1}{(2\pi)^{4-4\alpha}}\frac{2\pi^{2-2\alpha}}{\Gamma(2-2\alpha)}.$$

For the anomalous dimension of the composite operator $O_2\equiv\nabla\phi\nabla\phi$ we obtain (since the anomalous dimension of the field $\gamma_\phi=0$ in this case)

$$\gamma_{O_2}=\gamma_2(\lambda_*)=\frac{3(n+2)(1-2\alpha)}{2(n+8)^2(1-\alpha)}\tilde{\varepsilon}^2 \tag{53}$$

where λ_* is the non-trivial fixed point of the beta function (52) and the standard definition

$$\gamma_2=-\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_2$$

has been used. This composite operator should become marginal or relevant in order to render the long-range fixed point unstable, which means that its total dimension should become non-positive. However, the anomalous dimension of the operator $O_2=\nabla\phi\nabla\phi$ turned out to be positive, resulting in the total dimension

$$d_{O_2}=2\alpha+\gamma_{O_2}=2\alpha+\frac{3(n+2)(1-2\alpha)}{2(n+8)^2(1-\alpha)}\tilde{\varepsilon}^2 \tag{54}$$

which is positive for small positive α and thus bears no indication that the LR regime would become unstable at $\alpha>0$, contrary to the results obtained above! Moreover, in the treatment of preceding sections the relation $\alpha=O(\varepsilon)$ is assumed to hold and therefore in this scheme the SR term is always relevant, owing to the very construction of the renormalisation procedure. However, the SR term is obviously irrelevant for finite $\alpha>0$, which correspond to the purely LR model whereas, due to (54), there seems to be no way to describe the crossover from the purely LR regime to the 'mixed' regime described by the fixed point (27). On the other hand, the two approaches complement each other in a consistent way in the problem of diffusion in a random environment [12]. Thus, the relation between these approaches to the crossover problem appears to be somewhat controversial. Unfortunately, in the random disorder model (32) there is no infrared stable fixed point for finite $\xi>0$ corresponding to a purely LR regime and therefore the discrepancy cannot be tested in this model.

6. Conclusion

In this paper, field-theoretic renormalisation group techniques have been applied to the analysis of the interplay between short-range and long-range exchange (correlations) in the ϕ^4 model, both with a temperature-like quenched disorder and without it. In both cases anomalous dimensions of fields and composite operators are shown to be continuous functions of the parameters ξ (α) characterising the power-like falloff $1/r^{d-2\xi}(1/r^{d+2-2\alpha})$ of the long-range correlations (exchange). Critical exponents are shown to assume their short-range values for $\xi \leq d/2-1/\nu$ ($\alpha \leq \eta/2$), where ν is the correlation length exponent of the ϕ^4 model with short-range correlated disorder and η is the Fisher exponent of the 'pure' ϕ^4 model. The validity of the scaling law of Weinrib and Halperin [2]: $(d-2\xi)\nu = 2$, as well as the Harris criteria [2, 13] has been confirmed to all orders in perturbation theory. A controversy between the approaches of Weinrib and Halperin [2] and Amit and Peliti [10] to the interplay of long-range and short-range exchange in the ϕ^4 model is pointed out.

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